Scientia Iranica A (2012) 19 (3), 417–422

SCIENTIA IRANICA

Sharif University of Technology

Scientia Iranica

Transactions A: Civil Engineering

www.sciencedirect.com

Coupling of homotopy and the variational approach for a conservative oscillator with strong odd-nonlinearity

Y. Khan [a,*], **M. Akbarzade** [b], **A. Kargar** [b]

[a] *Department of Mathematics, Zhejiang University, Hangzhou 310027, China*
[b] *Department of Mechanical Engineering, Quchan Branch, Islamic Azad University, Quchan, Iran*

Received 4 January 2012; revised 18 January 2012; accepted 14 February 2012

KEYWORDS
Odd-nonlinearity;
Cubic–quintic duffing equation;
Natural frequency;
Homotopy perturbation method;
Variational approach.

Abstract In this paper, a new approach combining the features of the homotopy concept with the variational approach is proposed for describing and predicting analytical approximations of a conservative oscillator with strong odd-nonlinearity. The new technique does not depend upon small parameter assumptions, and incorporates salient features of both methods of homotopy perturbation and the variational approach. The cubic–quintic duffing oscillator is analyzed to illustrate the usefulness and effectiveness of the proposed technique. Four approximate formulas for the frequency are established for small, as well as large, amplitudes of motion. The results of applying this procedure to the cubic–quintic duffing equation are compared to those analytical and exact solutions, in order to substantiate the accuracy and correctness of the approximate analytical approach. The approach can be easily extended to other nonlinear oscillators.

1. Introduction

Nonlinear oscillator models have been widely used in many areas, and their significant importance is not just limited to physics and engineering. Mechanical oscillatory systems are often named governed nonlinear differential equations by different authors [1–5].

In general, such problems are not amenable to exact treatment. The Nayfeh perturbation method [6], involving expansion over a small parameter (perturbation quantity), has been the most common analytical technique for nonlinear oscillation systems. In general, it is only useful if small parameters exist in the nonlinear systems, where the solution can be analytically expanded into a power series of parameters. Many nonlinear problems do not contain such perturbation quantity, so, in order to overcome the shortcomings, many new techniques have appeared in open literature, such as the variational iteration method [7–9], the energy balance method [10–12], the Hamiltonian approach [13,14], the variational approach [15–17], amplitude–frequency formulation [18] and other classical methods [19–27].

In this paper, a new approach, combining the features of the homotopy concept with the variational approach, is proposed to find accurate analytical solutions for nonlinear cubic-quintic duffing equations. The coupled method of He's homotopy perturbation method [19] and variational formulation [15], couples homotopy with the variational appproach. The method first constructs a homotopy equation, and, then, the solution is expanded into a series of p. As the zeroth order approximate solution is easy to obtain, the second term is solved using the variational approach, where the frequency of the nonlinear oscillator can be obtained. This technique is very similar to Marinca's work [21], where the unknown parameters are identified using least squares technology.

The Duffing equation is a well-known nonlinear differential equation, which is related to many practical engineering systems, such as the classical nonlinear spring system, with odd nonlinear restoring characteristics, and more recently in different physical phenomena (see [2]). The unperturbed cubic-quintic duffing equation can be found in the modeling of the

* Corresponding author.
E-mail addresses: yasir123@zju.edu.cn (Y. Khan), mehdiakbarzade@yahoo.com (M. Akbarzade).
Peer review under responsibility of Sharif University of Technology.

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doi:10.1016/j.scient.2012.04.004

free vibration of a restrained uniform beam carrying an intermediate lumped mass and undergoing large amplitudes of oscillation in the unimodel duffing type temporal problem [28], the nonlinear dynamics of a slender elastica, the generalized Pochhammer–Chree (PC) equation and the compound Korteweg-de Vries (KdV) equation [28].

Due to the presence of strong nonlinearities, the accuracy of approximate analytical methods becomes extremely demanding. Lai et al. [28], Ganji et al. [29] and Pirbodaghi et al. [30] have obtained approximations of the cubic–quintic duffing equation by using different methods.

A cubic-quintic duffing oscillator of a conservative autonomous system can be described by the following differential equation, with cubic-quintic nonlinearities [28]:

$$u'' + \alpha u + \beta u^3 + \gamma u^5 = 0, \tag{1}$$

with initial conditions:

$$u(0) = A, \qquad u'(0) = 0. \tag{2}$$

It is a simple harmonic oscillator if $\alpha \neq 0$, $\beta = \gamma = 0$, a cubic Duffing oscillator if $\beta \neq 0$, $\gamma = 0$, and a quintic oscillator if $\gamma \neq 0$, $\beta = 0$. Otherwise, it is a cubic-quintic oscillator if β and γ do not vanish.

2. Application of the coupled homotopy-variational formulation

In this paper, a coupling method of the He's homotopy technique [19] and the varitional approach [15] is proposed to solve non linear oscillation systems. In contrast to traditional perturbation methods, the proposed method does not require a small parameter in the equation. In this method, according to the homotopy technique, a homotopy with an imbedding parameter, $p \in [0, 1]$, is constructed, as the zeroth order approximate solution is easy to be obtained. The second term is solved using the variational approach, where the frequency of the nonlinear oscillator can be obtained.

By considering the nonlinear oscillator, Eq. (1), the following homotopy can be constructed:

$$u'' + \omega^2 u + p\left[\beta u^3 + \gamma u^5 + (\alpha - \omega^2)u\right] = 0, \quad p \in [0, 1], \tag{3}$$

when $p = 0$, Eq. (3) becomes the linearized equation, $u'' + \omega^2 u = 0$. When $p = 1$, it turns out to be the original one. Assume that the periodic solution to Eq. (1) may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots. \tag{4}$$

Substituting Eq. (4) into Eq. (3), and collecting terms of the same power of p, gives:

$$u_0'' + \omega^2 u_0 + = 0, \tag{5}$$

$$u_1'' + \omega^2 u_1 + \beta u_0^3 + \gamma u_0^5 + (\alpha - \omega^2)u_0 = 0. \tag{6}$$

The solution of Eq. (5) is $u_0 = A\cos\omega_1 t$, where ω_1 will be identified from the variational formulation foru_1, which reads:

$$J(u_1) = \int_0^T \left\{ -\frac{1}{2}u_1'^2 + \frac{1}{2}\omega_1^2 u_1^2 + \beta u_0^3 u_1 + \gamma u_0^5 u_1 + (\alpha - \omega_1^2)u_0 u_1 \right\} dt, \quad T = \frac{2\pi}{\omega}. \tag{7}$$

2.1. First-order analytical approximation

To better illustrate the procedure, a simple trail function can be chosen:

$$u_1 = B_1(\cos\omega_1 t - \cos 3\omega_1 t). \tag{8}$$

Substituting u_1 into functional Eq. (7) results in:

$$J(A, B_1, \omega_1) = \frac{\pi B_1}{\omega_1}\left(\frac{1}{2}\beta A^3 + \alpha A - 4\omega_1^2 B_1 - A\omega_1^2 + \frac{5}{16}\gamma A^5\right). \tag{9}$$

Setting [15]:

$$\frac{\partial J}{\partial B_1} = 0, \qquad \frac{\partial J}{\partial \omega_1} = 0. \tag{10}$$

The solutions of Eq. (10) are:

$$B_1 = \frac{A}{\omega_1^2}\left(-\frac{1}{8}\omega_1^2 + \frac{5}{128}\gamma A^4 + \frac{1}{16}\beta A^2 + \frac{1}{8}\alpha\right), \tag{11}$$

$$\omega_1 = \sqrt{\frac{5}{16}\gamma A^4 + \frac{1}{2}\beta A^2 + \alpha}, \tag{12}$$

where angular frequency, ω_1, is the first-order analytical approximation.

2.2. Second-order analytical approximation

The accuracy of the first-order approximate solution can be dramatically improved if the trail function is chosen:

$$u_1 = B_1(\cos\omega_2 t - \cos 3\omega_2 t) + B_3(\cos 3\omega_2 t - \cos 5\omega_2 t). \tag{13}$$

Substituting Eq. (13) into Eq. (9) leads to the result:

$$J(A, B_1, B_3, \omega_2) = \frac{\pi}{4\omega_2}\left(\gamma B_3 + \frac{5}{4}\gamma B_1\right)A^5 + \frac{\pi}{2\omega_2}\left(\frac{1}{2}\beta B_3 + \beta B_1\right)A^3 + \frac{\pi B_1}{\omega_2}\left(\alpha - \omega_2^2\right)A + 4\pi\omega_2\left(2B_1B_3 - B_1^2 - 4B_3^2\right). \tag{14}$$

The stationary condition of Eq. (14) requires that [15]:

$$\frac{\partial J}{\partial B_1} = 0, \qquad \frac{\partial J}{\partial B_3} = 0, \qquad \frac{\partial J}{\partial \omega} = 0. \tag{15}$$

The solutions of Eq. (15) are:

$$B_3 = \frac{\gamma}{128\omega_2^2}A^5 + \frac{\beta}{128\omega_2^2}A^3 + \frac{1}{4}B_1, \tag{16}$$

$$B_1 = \frac{\gamma}{16\omega_2^2}A^5 + \frac{3\beta}{32\omega_2^2}A^3 + \frac{(\alpha - \omega_2^2)}{6\omega_2^2}A. \tag{17}$$

The second-order approximate frequency can be obtained as follows: (see Box I) where angular frequency, ω_2, is the second-order analytical approximation.

2.3. Third-order analytical approximation

To construct the third-order analytical approximations, assume the trail function is:

$$\omega_2 = \sqrt{-\left(\frac{3}{8}A^4\gamma + \frac{9}{16}A^2\beta + \alpha\right) + \sqrt{\frac{153}{256}A^8\gamma^2 + \frac{225}{128}A^6\beta\gamma + \left(\frac{333}{256}\beta^2 + 3\gamma\alpha\right)A^4 + \frac{9}{2}A^2\beta\alpha + 4\alpha^2}}, \tag{18}$$

Box I:

$$u_1 = B_1(\cos\omega_2 t - \cos 3\omega_2 t) + B_3(\cos 3\omega_2 t - \cos 5\omega_2 t) + B_5\left(3\cos 5\omega_2 t - \frac{5}{3}\cos 7\omega_2 t\right). \tag{19}$$

Substituting Eq. (19) into Eq. (9) leads to the result:

$$J(A, B_1, B_3, B_5, \omega_3) = \frac{\pi}{4\omega_3}\left(\gamma B_3 + \frac{3}{4}\gamma B_5 + \frac{5}{4}\gamma B_1\right)A^5 + \frac{\pi}{2\omega_3}\left(\frac{1}{2}\beta B_3 + \beta B_1\right)A^3 + \frac{\pi B_1}{\omega_3}(\alpha - \omega_3^2)A + 4\pi\omega_3\left(18B_3B_5 + 2B_1B_3 - B_1^2 - 4B_3^2 - \frac{131}{3}B_5^2\right). \tag{20}$$

The stationary condition of Eq. (20) requires that [15]:

$$\frac{\partial J}{\partial B_1} = 0, \quad \frac{\partial J}{\partial B_3} = 0, \quad \frac{\partial J}{\partial B_5} = 0, \quad \frac{\partial J}{\partial \omega} = 0 \tag{21}$$

The solutions of Eq. (21) are:

$$B_5 = \frac{9\gamma}{16768\omega_3^2}A^5 + \frac{27}{131}B_3, \tag{22}$$

$$B_3 = \frac{605\gamma}{35968\omega_3^2}A^5 + \frac{131\beta}{8992\omega_3^2}A^3 + \frac{131}{281}B_1, \tag{23}$$

$$B_1 = \frac{67\gamma}{640\omega_3^2}A^5 + \frac{231\beta}{1600\omega_3^2}A^3 + \frac{281(\alpha - \omega_3^2)}{1200\omega_3^2}A. \tag{24}$$

The third-order approximate frequency can be obtained as follows: (see Box II) where angular frequency, ω_3, is the third-order analytical approximation.

2.4. Fourth-order analytical approximation

To construct the fourth -order analytical approximations, assume the trail function is:

$$u_1 = B_1(\cos\omega_2 t - \cos 3\omega_2 t) + B_3(\cos 3\omega_2 t - \cos 5\omega_2 t) + B_5\left(3\cos 5\omega_2 t - \frac{5}{3}\cos 7\omega_2 t\right) + B_7\left(\frac{5}{3}\cos 7\omega_2 t - \frac{5}{7}\cos 9\omega_2 t\right). \tag{26}$$

Substituting Eq. (26) into Eq. (9) leads to the result:

$$J(A, B_1, B_3, B_5, B_7, \omega_4) = \frac{\pi}{4\omega_4}\left(\gamma B_3 + \frac{3}{4}\gamma B_5 + \frac{5}{4}\gamma B_1\right)A^5 + \frac{\pi}{2\omega_4}\left(\frac{1}{2}\beta B_3 + \beta B_1\right)A^3 + \frac{\pi B_1}{\omega_4}(\alpha - \omega_4^2)A + 4\pi\omega_4\left(\frac{100}{3}B_5B_7 + 18B_3B_5 + 2B_1B_3 - B_1^2 - 4B_3^2 - \frac{131}{3}B_5^2 - \frac{3200}{147}B_7^2\right). \tag{27}$$

The stationary condition of Eq. (27) requires that [15]:

$$\frac{\partial J}{\partial B_1} = 0, \quad \frac{\partial J}{\partial B_3} = 0, \quad \frac{\partial J}{\partial B_5} = 0, \quad \frac{\partial J}{\partial B_7} = 0, \quad \frac{\partial J}{\partial \omega} = 0. \tag{28}$$

The solutions of Eq. (28) are:

$$B_7 = \frac{49}{64}B_5, \tag{29}$$

$$B_5 = \frac{3\gamma}{3656\omega_4^2}A^5 + \frac{288}{989}B_3, \tag{30}$$

$$B_3 = \frac{1205\gamma}{43648\omega_4^2}A^5 + \frac{989\beta}{43648\omega_4^2}A^3 + \frac{989}{1364}B_1, \tag{31}$$

$$B_1 = \frac{97\gamma}{400\omega_4^2}A^5 + \frac{1239\beta}{4000\omega_4^2}A^3 + \frac{341(\alpha - \omega_3^2)}{750\omega_3^2}A. \tag{32}$$

The fourth-order approximate frequency can be obtained as follows: (see Box III) where angular frequency, ω_4, is the fourth-order analytical approximation.

3. Results and discussion

To show the efficiency of the presented method for cubic–quintic Duffing oscillators, in comparison with other results and the exact result, three cases for $\alpha = \beta = \gamma = 1, \alpha = 5, \beta = 3, \gamma = 1$ and $\alpha = 1, \beta = 10, \gamma = 100$ are given.

The relative errors of frequencies are defined as Lai et al. [28]:

$$Error(\%) = \frac{|\omega_i - \omega_{Exact}|}{\omega_{Exact}} \times 100, \quad i = 1, 2, 3, 4. \tag{34}$$

For reference, the exact frequency, ω_{Exact}, is obtained by direct integration of governing Eq. (1) of the dynamical system. Imposing initial conditions, the solution is [4]:

$$\omega_{exact}(A) = \frac{\pi k_1}{2\int_0^{\frac{\pi}{2}}\left(1 + k_2\sin^2\theta + k_3\sin^4\theta\right)^{-\frac{1}{2}}dt}, \tag{35}$$

where:

$$k_1 = \sqrt{\alpha + \beta\frac{A^2}{2} + \gamma\frac{A^4}{3}}, \tag{36}$$

$$k_2 = \frac{3\beta A^2 + 2\gamma A^4}{6\alpha + 3\beta A^2 + 2\gamma A^4}, \tag{37}$$

$$k_3 = \frac{2\gamma A^4}{6\alpha + 3\beta A^2 + 2\gamma A^4}. \tag{38}$$

Based on Eqs. (12), (18), (25) and (33), the various approximations compared, with respect, to the exact solution are:

$$\omega_3 = \sqrt{-\left(\frac{1005}{2248}A^4\gamma + \frac{693}{1124}A^2\beta + \alpha\right) + \sqrt{\frac{2180475}{256752}A^8\gamma^2 + \frac{2921985}{1263376}A^6\beta\gamma + \left(\frac{989973}{631688}\beta^2 + \frac{1005}{281}\gamma\alpha\right)A^4 + \frac{1386}{281}A^2\beta\alpha + 4\alpha^2}}, \quad (25)$$

Box II:

$$\omega_4 = \sqrt{-\left(\frac{1455}{2725}A^4\gamma + \frac{3717}{5456}A^2\beta + \alpha\right) + \sqrt{\frac{35552025}{29767936}A^8\gamma^2 + \frac{44621505}{14883968}A^6\beta\gamma + \left(\frac{56376981}{29767936}\beta^2 + \frac{1455}{341}\gamma\alpha\right)A^4 + \frac{3717}{682}A^2\beta\alpha + 4\alpha^2}}, \quad (33)$$

Box III:

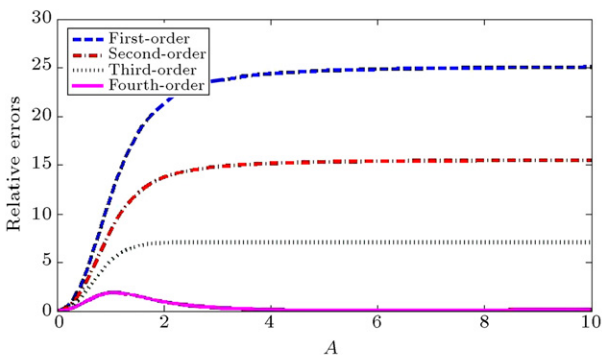

Figure 1: The relative errors for the approximate frequencies ($\alpha = \beta = \gamma = 1$).

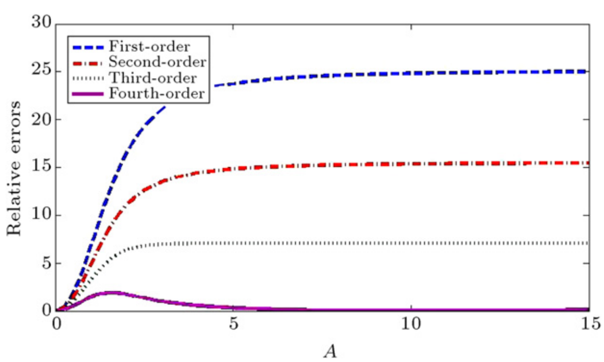

Figure 2: The relative errors for the approximate frequencies ($\alpha = 5, \beta = 3, \gamma = 1$).

$$\lim_{A\to\infty} \frac{\omega_1}{\omega_{Exact}} = 0.74851,$$
$$\lim_{A\to\infty} \frac{\omega_2}{\omega_{Exact}} = 0.84481,$$
$$\lim_{A\to\infty} \frac{\omega_3}{\omega_{Exact}} = 0.92950, \quad (39)$$
$$\lim_{A\to\infty} \frac{\omega_4}{\omega_{Exact}} = 1.00154.$$

The relative error of the fourth-order analytical approximation compared with the exact solution is less than 0.16% in the limit as $A \to \infty$. In Figures 1–3, the relative errors for the approximate frequencies for different parameters are plotted.

Lai et al. [28] obtained an analytic approximation of the cubic–quintic duffing equation by using the Newton harmonic balancing method:

$$\lim_{A\to\infty} \frac{|\omega_{Lai} - \omega_{Exact}|}{\omega_{Exact}} \times 100 = 0.23\%. \quad (40)$$

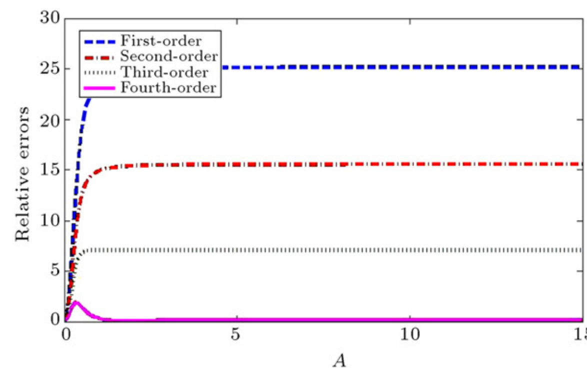

Figure 3: The relative errors for the approximate frequencies ($\alpha = 1, \beta = 10, \gamma = 100$).

Also, Ganji et al. [29] obtained another analytic approximation of the cubic–quintic duffing equation, using the energy balance method:

$$\lim_{A\to\infty} \frac{|\omega_{Ganji} - \omega_{Exact}|}{\omega_{Exact}} \times 100 = 2.29\%. \quad (41)$$

Hence, the proposed method is suitable for solving Eq. (1) with any large amplitude of oscillation, A. The fourth-order analytical approximation is more accurate than the results of Lai et al. [28] and Ganji et al. [29], when $A \to \infty$. The fourth-order analytical approximation are given and compared, analytically, with the results of Ganji et al. [29], Lai et al. [28] and the exact one, for different oscillation amplitudes in Tables 1–3, respectively.

4. Conclusions

A coupled homotopy-variational formulation has been applied to obtain analytical approximate solutions for a nonlinear cubic-quintic duffing equation, which are conservative and periodic. This procedure is explicit, effective, and has a distinct advantage over usual approximation methods, in that, the approximate solution obtained here is valid, not only for weakly nonlinear equations, but, also, for strongly nonlinear ones. In particular, it would be desirable to determine easier ways of constructing trial functions for some complex nonlinear problems. The relative error is smaller for the fourth order coupled homotopy-variational formulation than that in Ganji and Lai's work [29,28]. In coupled homotopy-variational formulation, the higher-order approximates can readily be obtained with high accuracy. It is obvious that the variational approach provides us with a freedom of choice of trial function, and gives us more information regarding the relation between frequency and amplitude. Convergence and error studies for the above mentioned method are further needed, and it is clear that many modifications can be made.

Table 1: Comparison of current frequency and existing results for cubic–quintic Duffing oscillator and $\alpha = \beta = \gamma = 1$.

A	ω_{Exact}	ω_1	ω_2	ω_3	ω_4	Lai et al. [28]	Ganji et al. [29]
0.1	1.0037770	1.0025125	1.0028276	1.0031009	1.0034276	1.00377	1.00377306
0.5	1.1065487	1.0698277	1.0792589	1.0877056	1.0974873	1.10655	1.10635650
1	1.5235914	1.3462912	1.3984287	1.4456576	1.4951413	1.52375	1.10635650
5	19.1815720	14.4503460	16.2514248	17.8276787	19.1806374	19.2215	19.608880
10	75.1776276	56.3560104	63.5547600	69.8760834	75.2651825	75.3454	76.889585
20	299.22427	224.05580	252.83170	278.12708	299.65771	299.903	306.06681
50	1 867.5796	1 397.9900	1 577.7996	1 735.9103	18 704.304	1 871.84	1 910.33222
100	7 468.8525	5 590.6172	6 309.8315	6 942.2827	7 480.3364	7 485.89	7 639.85509
500	186 709.59	139 754.70	157 734.86	173 546.20	186 997.34	187 135.59	190 984.592
1 000	746 836.94	559 017.44	630 938.13	694 183.44	747 988.00	748 540.91	763 936.894

Table 2: Comparison of current frequency and existing results for cubic–quintic Duffing oscillator and $\alpha = 5$, $\beta = 3$, $\gamma = 1$.

A	ω_{Exact}	ω_1	ω_2	ω_3	ω_4	Lai et al. [28]	Ganji et al. [29]
0.1	2.2411156	2.2394266	2.2398469	2.2402105	2.2406456	2.2411	2.241102478
0.5	2.3661575	2.3226130	2.3337476	2.3434565	2.3548751	2.36615	2.366246867
1	2.7962794	2.6100767	2.6612775	2.7063482	2.7566507	2.79630	2.798963393
5	20.2164536	15.4211702	17.2236977	18.7895069	20.1565380	20.2514	20.64011142
10	76.1700134	57.2712860	64.4817429	70.7962723	76.2022247	76.3326	77.88483819
50	1 868.5568	1 398.8853	1 578.7106	1 736.8159	1 871.3540	1 872.81	1 911.314776
100	7 469.8296	5 591.5117	6 310.7422	6 943.1880	7 481.2598	7 486.86	7 640.837246
500	186 710.58	139 755.59	157 735.78	173 547.11	186 998.27	187 136.56	190 985.5701
1 000	746 837.94	559 018.31	630 939.00	694 184.38	747 988.88	748 541.88	763 937.8765

Table 3: Comparison of current frequency and existing results for cubic–quintic Duffing oscillator and $\alpha = 1$, $\beta = 10$, $\gamma = 100$.

A	ω_{Exact}	ω_1	ω_2	ω_3	ω_4	Lai et al. [28]	Ganji et al. [29]
0.1	1.0397019	1.0262188	1.0296117	1.0325994	1.0361322	1.03970	1.039642196
0.5	2.5247023	2.0501525	2.2114758	2.3542032	2.4890764	2.52642	2.554014562
1	8.0100698	6.1032777	6.8193183	7.4440041	7.9883609	8.02429	8.176911017
5	187.19966	140.20432	158.19193	174.00040	187.46034	187.623	191.4770915
10	747.32526	559.46490	631.39343	694.63605	748.44946	749.027	764.4279087
50	18 671.400	13 975.872	15 773.896	17 355.027	18 700.148	18 714.00	19 098.90111
100	74 684.133	55 902.148	63 094.219	69 418.750	74 799.211	74 854.53	76 394.13136
500	1 867 091.6	1 397 542.9	1 577 344.5	1 735 457.9	1 869 969.3	1 871 351.54	1 909 841.499
1 000	7 468 365.0	5 590 170.5	6 309 377.0	6.941 830.5	7 479 875.5	7 485 404.69	7 639 364.525

Acknowledgments

The authors are very grateful to the Department of Mechanical Engineering at the Quchan Branch of the Islamic Azad University for the provision of excellent research facilities.

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Y. Khan is editor of two International Journals. His current research mainly covers approximate, analytical and numerical solutions of nonlinear problems arising in applied sciences and engineering phenomena.

M. Akbarzade obtained his M.S. degree in Mechanical Engineering from Mazandaran University, Iran, and has, since 2008, been working as lecturer in the Islamic Azad University. His main field of research is focused on nonlinear dynamics and vibration and he has published more than 18 papers in this subject area.

A. Kargar is a Ph.D. student in Mechanical Engineering from Ferdowsi University, Mashahd, Iran, and became a faculty member of the Islamic Azad University in 2005. His main field of interest is focused on fluid dynamics and numerical methods.